Lecture Notes on Continuum Mechanics of Solids(P402), M.Sc. Physics

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Unit 1: Continuum Mechanics of Solid Media -

Notes

Review of Cartesian Tensors:

Understanding Tensors: Qualitative & Quantitative approaches to understand tensors: a mathematical object which remains invariant under a coordinate transformation. Such objects are defined by their transformation properties. Tensors also are the generalisation of scalars and vectors.

Object	No. of magnitudes	No. of direction senses	No. of components (3^n)	Rank (n)
Scalar	1	0	1	0
Vectors	1	1	3	1
Dyads	1	2	9	2
Triads	1	3	27	3

And so on.

Tensors are characterised by their **invariance under coordinate transformations**.

Identifying Tensors in physics

1. Stress Tensor:
   The expression of stress (T),
   dF = **T** . **dS ,** dF and dS are force and surface elements respectively. **T** is a rank 2 tensor, to which dS is multiplied using the inner product.

2. Magnetic Flux Density:
   In the relation of Magnetic Field **B** with Magnetization **H**
   B = **μ̲** . **H** , **μ̲** is the permeability tensor of rank 2.
3. Angular Momentum
   In the definition of Spin angular momentum L,
   L= **I̲** . ω, **I̲** is the moment of inertia tensor.

Contraction of a rank using a tensor - Inner Product
Tensors can be multiplied to other tensors using inner (dot) products. Under such multiplication, the rank of the product reduces by 2.
Let **A** and **B** be two tensors of ranks n & m respectively.
A.B = **C,** another tensor whose rank is equal to (n+m-2).

<u>Suffix Notation and the Summation Convention</u>

We will consider vectors in 3D, though the notation we shall introduce applies (mostly) just as well to n dimensions. For a general vector

$$x = (x^1, x^2, x^3)$$

we will refer to x^i , the i th component of x.
The index i may take any of the values 1, 2 or 3, and we refer to

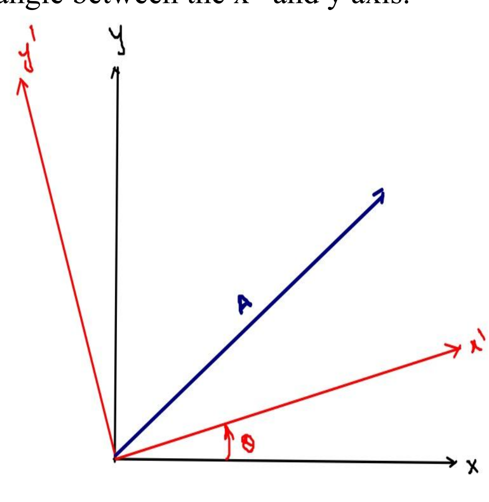
$$\text{vector } x^i = \text{ the vector whose components are } (x^1, x^2, x^3) .$$

However, we cannot write x = x^i , since the LHS is a vector and the RHS a scalar. Instead, we can write $[x]^i = x^i$, and similarly $[x + y]^i = x^i + y^i$.
The expression $x^i = y^i$ implies that y = x; the statement in suffix notation is implicitly true for all three possible values of i (1,2,3).

<u>Transformation of a vector</u>
Let {x ′ , y′} axes be different from {x, y} by a simple rotation. Then the components of a vector A in the two coordinate systems are related by

$$A'_x = A_x \cos(x', x) + A_y \cos(x', y)$$
$$A'_y = A_x \cos(y', x) + A_y \cos(y', y),$$

where (x ′ , y) denotes the angle between the x ′ and y axis.



Using the index notation, it can also be written

$$A'_1 = A_1 \cos(x'_1, x_1) + A_2 \cos(x'_1, x_2)$$
$$A'_2 = A_1 \cos(x'_2, x_1) + A_2 \cos(x'_2, x_2)$$

In three-dimensions, the components of a vector are transformed under rotation as follows

$$\mathbf{A'_1 = A_1 \cos(x'_1, x_1) + A_2 \cos(x'_1, x_2) + A_3 \cos(x'_1, x_3)}$$
$$\mathbf{A'_2 = A_1 \cos(x'_2, x_1) + A_2 \cos(x'_2, x_2) + A_3 \cos(x'_2, x_3)}$$
$$\mathbf{A'_3 = A_1 \cos(x'_3, x_1) + A_2 \cos(x'_3, x_2) + A_3 \cos(x'_3, x_3)}$$

Converting this in a matrix,

$$\begin{pmatrix} A_1' \\ A_2' \\ A_3' \end{pmatrix} = \begin{pmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix}$$

$$C_{ij} = \cos(x_i', x_j)$$

Orthogonality of Cik

Since the length of vector A must be invariant, i.e., the same in both coordinate systems,

$$\mathbf{(A'_i).(A'_i) = (A_i).(A_i)}$$

$$\mathbf{(A'_i).(A'_i) = [C_{ik}](A_k)[C_{ij}](A_j) = [C_{ik}][C_{ij}](A_k)(A_j) = (A_j)(A_j)} .$$

Introducing the Kronecker delta δ_{kj} we have,

$$[C_{ik}][C_{ij}] = \delta_{kj}$$

This property is called the orthogonality of the transformation matrix C, which is a generalization of a similar concept in vectors.

Contra and Covariant Transformations

The Transformation matrix used above [Cij] is of two types, in general:

1. Contravariant
2. Covariant

Contravariant type transformation:

$$\begin{pmatrix} x^{1'} \\ x^{2'} \end{pmatrix} = \begin{pmatrix} \text{proj. of } x^{1'} \text{ on } x^1 & \text{proj. of } x^{1'} \text{ on } x^2 \\ \text{proj. of } x^{2'} \text{ on } x^1 & \text{proj. of } x^{2'} \text{ on } x^2 \end{pmatrix} \begin{pmatrix} x^1 \\ x^2 \end{pmatrix}$$

$$x^{\mu'} = C_{\mu\nu} x^{\nu}, \quad C_{\mu\nu} = \text{proj. of } x^{\mu'} \text{ on } x^{\nu}$$

Covariant type transformation:

$$\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{pmatrix} \text{proj. of } x^1 \text{ on } x^{1'} & \text{proj. of } x^2 \text{ on } x^{1'} \\ \text{proj. of } x^1 \text{ on } x^{2'} & \text{proj. of } x^2 \text{ on } x^{2'} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$x_{\mu}' = \left(C_{\mu\nu}^{-1}\right)^T x_{\nu} \quad ; \quad \left(C_{\mu\nu}^{-1}\right)^T = \text{proj. of } x_{\nu} \text{ on } x_{\mu}'$$

Definition of a Cartesian Tensor

- A tensor T of rank n is an array of components denoted by Tijk...m with n indices ijk . . . m.
- In three dimensional space T has 3^n components.
- The defining property of a cartesian tensor is the following law : From coordinate system S to S ′ by a rotation, the components of a tensor transform according to
  T ' ijk...m =[Cis][Cjt][Cku] · · · [Cmv]Tstu...v.
- As special cases, a scalar is a zero-th rank tensor T ′ = T.
- A vector is a first rank tensor which is transformed according to
  T′ i = ‘Cij].Tj
- A second rank tensor is transformed according to
  T ' ij = [Cis][Cjt]Tst.

The Quotient Law

A set of 3^n *numbers form the components of a tensor of rank n, if and only if its scalar product with another arbitrary tensor is again a tensor*.

This is called the quotient law and can be used to check whether a set of numbers form a tensor.

Tensor Algebra

(a) Addition: The sum of two tensors of equal rank is another tensor of the same rank.

(b) Multiplication. (A tensor of rank b) times (a tensor of rank c) = a tensor of rank b + c with 3b+c components Eij...krs...t = Aij...kBrs...t.

(c) Contraction : If any pair of indices of an n-th rank tensor are set equal and summed over the range 1,2,3, the result is a tensor of rank n− 2.
Consider the relation
dF = **T** . **dS**
dF is a vector i.e. a tensor of rank 1
T is a dyad i.e. a tensor of rank 2
dS is a vector i.e. a tensor of rank 2
Rank of dF = (Rank of T)+ (Rank of dS) -2 (Since T.dS is a dot product)
1 = 2 + 1 - 2
Thus a scalar product is the result of multiplication and contraction

Summary:
1. All scalars are not tensors, although all tensors of rank 0 are scalars
2. All vectors are not tensors, although all tensors of rank 1 are vectors

3. All dyads or matrices are not tensors, although all tensors of rank 2 are dyads or matrices
4. The product of a tensor and a scalar (tensor of rank 0) is commutative. The pre-multiplication of a given tensor by another tensor produces a different result from post-multiplication; i.e., tensor multiplication in general is not commutative.
9. The rank of a new tensor formed by the product of two other tensors is the sum of their individual ranks.
10. The inner product of a tensor and a vector or of two tensors is not commutative.
11. The rank of a new tensor formed by the inner product of two other tensors is the sum of their individual ranks minus 2.
12. A tensor of rank n in three-dimensional space has 3n components.
13. Tensors are described by their transformation properties
14. Transformations are of two types - Contravariant and Covariant transformations